

Hermite–Padé approximation (basic notions and theorems)

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Abstract: Our purpose is to give a brief exposition of basic notions and facts on Hermite–Padé approximation theory. A Hermite–Padé form (HPF) is defined here in the same way as a Padé approximant is. As a consequence, in some cases HPF does not exist. Even in the modified version HPF need not be unique. Therefore it was worthwhile to choose some HPF as the optimal one.

All criteria of existence, unicity, etc. are phrased in a unified manner. They base on few quantities depending on the power series under consideration and are usable in the theory rather than in the numerical practice.

Keywords: Hermite–Padé approximation, Hermite–Padé form, optimal form, Hermite–Padé table.

1. Preliminaries

We will denote by ∂P the degree of a polynomial P ; in particular, $\partial P = -1$ iff $P \equiv 0$. For a power series $F(x) = \sum_{k=0}^{\infty} a_k x^k$, its order, denoted by $\text{ord } F$, is equal to k if $a_0 = a_1 = \dots = a_{k-1} = 0$, $a_k \neq 0$ and to $+\infty$ if $F \equiv 0$.

Definition 1.1. Let $s > 1$ be a natural number. Let s series

$$F_i(x) := \sum_{k=0}^{\infty} a_{ik} x^k, \quad i = 1, 2, \dots, s,$$

be given. A system $L = (l_1, l_2, \dots, l_s)$ of integers is such that $l_i \geq -1$, $i = 1, 2, \dots, s$, and at least one of them is nonnegative. A *Hermite–Padé form of degree L* is, by definition, every system

$$\Phi(L) \equiv \Phi(l_1, l_2, \dots, l_s) = (P_{L1}, P_{L2}, \dots, P_{Ls})$$

of polynomials such that

$$\partial P_{Li} \leq l_i, \quad i = 1, 2, \dots, s,$$

$$\text{ord} \sum_{i=1}^s F_i P_{Li} \geq \lambda(L), \quad \text{where } \lambda(L) := \sum_{i=1}^s l_i + s - 1, \quad (1)$$

$$\exists i \ P_{Li}(0) \neq 0. \quad (2)$$

$R\Phi(L)$ denotes the sum from (1) and is called the *rest* of $\Phi(L)$.

In the preceding papers on Hermite–Padé approximation the condition $P_{Li} \neq 0$ occurs instead of $P_{Li}(0) \neq 0$. We prefer the version (2) because in this case Hermite–Padé approximation is a strict generalization of Padé approximation (obtained for $s = 2$, $F_1 \equiv -1$, $F_2 = F$).

It is not our purpose to describe here any applications of Hermite–Padé approximation. It suffices to recall that in [1, Section 1.3] some particular cases of such an approximation were described; for a given power series

$$F(x) = \sum_{k=0}^{\infty} a_k x^k,$$

the so-called G^3J approximation ($s = 3$, $F_1 = F''$, $F_2 = F'$, $F_3 = F$) or Shafer quadratic approximation ($s = 3$, $F_1 = F^2$, $F_2 = F$, $F_3 \equiv 1$) may be used. Another particular case can be recommended when a type of principal singularity of F is known. If, for example, $a_k \sim ck^{-2}$, $c \neq 0$, then we can put

$$s = 3, \quad F_1(x) = -\frac{1-x}{x} \log(1-x), \quad F_2(x) \equiv -1, \quad F_3(x) = F(x).$$

Then (2) implies that

$$F(x) \approx \frac{1}{P_{L3}(x)} \left[\frac{1-x}{x} \log(1-x) P_{L1}(x) + P_{L2}(x) \right].$$

If for every L a form $\Phi(L)$ exists and is defined uniquely up to a constant factor, such forms can be calculated recursively. It seems that the most general numerical method working under this condition (in fact it may be weakened there) is given in [2]. This paper is motivated, in particular, by a wish to create such a theoretical base of Hermite–Padé approximation which will allow to find analogous methods working practically without any conditions.

Definition 1.2. For a degree L and a natural number μ let

$$\mathcal{A}(L, \mu) := \begin{bmatrix} a_{10} & & 0 & & a_{s0} & & 0 \\ a_{11} & & & & a_{s1} & & \\ \vdots & & a_{10} & & \vdots & & a_{s0} \\ \vdots & & \vdots & \cdots & \vdots & & \vdots \\ a_{1,\mu-1} & \cdots & a_{1,\mu-l_1-1} & & a_{s,\mu-1} & \cdots & a_{s,\mu-l_s-1} \end{bmatrix}$$

The matrix has μ rows and $\lambda(L) + 1$ columns divided into s (possibly empty) groups; the i th group is composed of $l_i + 1$ columns depending only on F_i . Let $\rho(L, \mu)$ denote the rank of matrix $\mathcal{A}(L, \mu)$. In addition, $\rho(L, 0) := 0$.

Lemma 1.3. For every L the sequence $\{\rho(L, \mu)\}$, $\mu = 0, 1, \dots$, is such that

$$0 \leq \rho(L, \mu + 1) - \rho(L, \mu) \leq 1, \quad \rho(L, \mu) \leq \lambda(L) + 1, \quad \mu = 0, 1, \dots$$

Definition 1.4. The system (F_1, F_2, \dots, F_s) of power series is called *polynomially dependent* if there exist polynomials P_1, P_2, \dots, P_s not all vanishing identically and such that

$$\sum_{i=1}^s F_i P_i \equiv 0;$$

otherwise it is called *polynomially independent*.

For the sake of simplicity, we shall assume that (F_1, F_2, \dots, F_s) is polynomially independent. This condition is not particularly restrictive and can be weakened or even omitted in most theorems.

Lemma 1.5. *For every system (F_1, F_2, \dots, F_s) and every degree $L = (l_1, l_2, \dots, l_s)$ the set of solutions of the linear homogeneous system*

$$\sum_{i=1}^s \sum_{j=0}^{\min\{k, l_i\}} a_{i,k-j} p_{ij} = 0, \quad k = 0, 1, \dots, \mu - 1, \quad (3)$$

in $\lambda(L) + 1$ unknowns $p_{10}, \dots, p_{1l_1}, \dots, p_{s0}, \dots, p_{sl_s}$ is of dimension $\lambda(L) + 1 - \rho(L, \mu)$. If $\rho(L, \mu) \leq \lambda(L)$ (in particular, if $\mu = \lambda(L)$), there exist polynomials

$$P_i(x) = \sum_{k=0}^{l_i} p_{ik} x^k, \quad i = 1, 2, \dots, s,$$

with coefficients p_{ik} satisfying (3), not all vanishing identically and such that

$$\text{ord} \sum_{i=1}^s F_i P_i \geq \mu.$$

Lemma 1.6. *There exists an integer $\omega(L)$ such that*

$$\omega(L) \geq \lambda(L), \quad \rho(L, \omega(L)) = \lambda(L), \quad \rho(L, \omega(L) + 1) = \lambda(L) + 1.$$

The numbers $\lambda(L)$, $\rho(L, \mu)$ and $\omega(L)$ are essential in studying the existence or uniqueness of a Hermite–Padé form (Sections 2 and 3, respectively), existence of an optimal Hermite–Padé form (Section 4) and a block structure of Hermite–Padé table (Section 5). It should be emphasized that we shall consider all these problems from a purely theoretical point of view.

2. Existence

Lemma 1.5 does not guarantee the existence of Hermite–Padé form $\Phi(L)$. Theorem 2.1 gives its existence criterion.

For a degree $L = (l_1, l_2, \dots, l_s)$ and for a natural number n we write

$$L_n := (l_{n1}, l_{n2}, \dots, l_{ns}), \quad \text{where } l_{ni} := \max\{l_i - n, -1\}, \quad i = 1, 2, \dots, s. \quad (4)$$

Theorem 2.1. *A Hermite–Padé form $\Phi(L)$ exists and does not exist iff*

$$\rho(L, \lambda(L)) - \rho(L_1, \lambda(L) - 1) < \lambda(L) - \lambda(L_1)$$

and

$$\rho(L, \lambda(L)) - \rho(L_1, \lambda(L) - 1) = \lambda(L) - \lambda(L_1),$$

respectively.

Proof. The coefficients of every system of polynomials $P_1^*, P_2^*, \dots, P_s^*$ such that

$$\partial P_i^* \leq l_{1i}, \quad i = 1, 2, \dots, s, \quad \text{ord} \sum_{i=1}^s F_i P_i^* \geq \lambda(L) - 1$$

satisfy the linear homogeneous system of equations with the matrix $\mathcal{A}(L_1, \lambda(L) - 1)$. Since

$$\partial(xP_i^*(x)) \leq l_i, \quad i = 1, 2, \dots, s, \quad \text{ord} \sum_{i=1}^s F_i(x)xP_i^*(x) \geq \lambda(L),$$

the set of solutions of this system is contained in the set \mathcal{S} of solutions of (3). Then

$$\lambda(L_1) + 1 - \rho(L_1, \lambda(L) - 1) \leq \lambda(L) + 1 - \rho(L, \lambda(L)).$$

The equality holds iff the set of systems $(xP_1^*(x), xP_2^*(x), \dots, xP_s^*(x))$ is identical with \mathcal{S} , i.e., iff no solution of (3) is a Hermite–Padé form of degree L . \square

Similarly one can prove some other theorems.

3. Uniqueness

There are two different kinds of sets of all Hermite–Padé forms for a fixed degree L . The first kind is such that the nonuniqueness of Hermite–Padé form is not essential.

Definition 3.1. Let, for an L , $\Phi(L)$ exist. If there exist polynomials P_1, P_2, \dots, P_s such that every solution $(P_{L1}, P_{L2}, \dots, P_{Ls})$ of system (3) for $\mu = \lambda(L)$ is equal to $(VP_1, VP_2, \dots, VP_s)$, where V is a polynomial, then Hermite–Padé approximation of degree L is called *unique*.

If Hermite–Padé approximation of degree L is unique, then an approximant of F_j , say, resulting from an approximate equality

$$\sum_{i=1}^s F_i P_{Li} \approx 0$$

is the same for every form $\Phi(L)$.

Example 3.2. Let $s = 3$,

$$F_1(x) = -1 + x^2 + x^5 - x^6 + x^7 + x^8 + x^9 + \dots,$$

$$F_2(x) = 1 + x - x^5 + x^7 + 2x^8 - x^9 + \dots,$$

$$F_3(x) = -x - x^2 + 2x^6 - x^7 + x^8 + 2x^9 + \dots$$

Then every Hermite–Padé form $\Phi(2, 3, 0)$ is equal to

$$(p_{10} + p_{11}x + p_{12}x^2, (1-x)(p_{10} + p_{11}x + p_{12}x^2), 0),$$

where $p_{10} \neq 0$. The form depends on three parameters p_{10}, p_{11}, p_{12} but Hermite–Padé approximation of degree $(2, 3, 0)$ is unique. The simplest Hermite–Padé form is $(1, 1-x, 0)$.

Theorem 3.3. If Hermite–Padé approximation of degree L is unique and if the polynomials P_1, P_2, \dots, P_s are defined as above, then every form of degree L is equal to $(VP_1, VP_2, \dots, VP_s)$, where V is any polynomial such that $V \neq 0$ and $\partial V \leq \lambda(L) - \rho(L, \lambda(L))$.

Theorem 3.4. *If $\Phi(L)$ exists, then Hermite–Padé approximation of degree L is unique iff*

$$\rho(L_n, \lambda(L)) = \lambda(L_n), \quad \text{where } n := \lambda(L) - \rho(L, \lambda(L)) \quad (5)$$

(cf. (4)).

Proof. If (5) holds, then the linear homogeneous system with the matrix $\mathcal{A}(L_n, \lambda(L))$ has a nontrivial solution (unique up to a common factor). In other words, there exist polynomials P_1, P_2, \dots, P_s such that

$$\partial P_i \leq l_{ni}, \quad i = 1, 2, \dots, s, \quad \text{ord} \sum_{i=1}^s F_i P_i \geq \lambda(L), \quad \exists i \ P_i \neq 0.$$

For every V such that $0 \leq \partial V \leq n$ the conditions

$$\partial(VP_i) \leq l_i, \quad i = 1, 2, \dots, s, \quad \text{ord} \sum_{i=1}^s F_i VP_i \geq \lambda(L), \quad \exists i \ VP_i \neq 0$$

hold. Therefore the coefficients of VP_1, VP_2, \dots, VP_s satisfy (3) for $\mu = \lambda(L)$. This system has no other solutions.

If Hermite–Padé approximation of degree L is unique, then by Theorem 3.3 there exists a particular form $\Phi(L) = (P_1, P_2, \dots, P_s)$ such that every form of degree L is equal to $(VP_1, VP_2, \dots, VP_s)$ where $\partial V \leq n$. The coefficients of components of this particular form satisfy the linear homogeneous system with the matrix $\mathcal{A}(L_n, \lambda(L))$. The set of solutions of this system is one-dimensional because another dimension contradicts existence of Hermite–Padé form or uniqueness of approximation. \square

4. Optimal form

One can easily find an example of the nonuniqueness of Hermite–Padé approximation.

Example 4.1. For the series from Example 3.2,

$$\begin{aligned} \Phi(1, 1, 1) &= (p_{10} + p_{11}x, p_{10} + p_{21}x, p_{10} - p_{11} + p_{21} + p_{11}x) \\ &= p_{10}(1, 1, 1) + p_{11}(x, 0, -1 + x) + p_{21}(0, x, 1), \end{aligned} \quad (6)$$

provided that $p_{10} \neq 0$ or $p_{10} - p_{11} + p_{21} \neq 0$. Polynomials p_i from Definition 3.1 do not exist and Hermite–Padé approximation of degree $(1, 1, 1)$ is not unique. Therefore, for different values of free parameters we obtain, in general, different approximants of F_j . This essential nonuniqueness of $\Phi(1, 1, 1)$ may be useful. In fact, every form is a linear combination of three particular forms given in (6). Their rests are equal to $x^6 + x^7 + 4x^8 + \dots$, $-x^6 + 2x^7 - x^8 + \dots$ and $x^6 - x^7 + 2x^8 + \dots$, respectively. The parameters p_{10}, p_{11}, p_{21} can be chosen for the coefficients of x^6 and x^7 in $R\Phi(1, 1, 1)(x)$ to be annihilated:

$$p_{10} - p_{11} + p_{21} = 0, \quad p_{10} + 2p_{11} - p_{21} = 0.$$

This system has the general solution $(p, -2p, -3p)$. For $p = 1$ we have $\Phi(1, 1, 1) = (1 - 2x, 1 - 3x, -2x)$ and $R\Phi(1, 1, 1)(x) = (4p_{10} - p_{11} + 2p_{21})x^8 + \dots = \mathcal{O}(x^9)$. The increasing of ord $R\Phi(L)$ in comparison to the requirement (1) gives often better approximants and justifies

the choice of some particular form as the optimal one (Definition 4.2). On the other hand, the optimal form depends on additional coefficients of the series F_i , which can make its calculation troublesome.

Definition 4.2. It follows from the definition of $\omega(L)$ (cf. Lemma 1.6) that there exist polynomials P_i not all vanishing identically and such that

$$\partial P_i \leq l_i, \quad i = 1, 2, \dots, s, \quad \text{ord} \sum_{i=1}^s F_i P_i = \omega(L).$$

If $P_i(0) \neq 0$ for some i , then the system (P_1, P_2, \dots, P_s) is called the *optimal Hermite–Padé form of degree L* . It will be denoted by $\Phi_{\text{opt}}(L)$.

Corollary 4.3. If $\Phi_{\text{opt}}(L)$ exists, then it is a Hermite–Padé form having the rest of maximal (finite) order. $\Phi_{\text{opt}}(L)$ is determined uniquely up to a constant factor.

In some cases (whether or not Hermite–Padé approximation is unique) the optimal Hermite–Padé form of degree L does not exist, although the set of forms $\Phi(L)$ is not empty.

Example 4.4. For the series from Example 3.2,

$$\begin{aligned} \Phi(2, 2, 2) &= (p_{10} + p_{11}x + p_{12}x^2, p_{10} + (-p_{10} + p_{11})x + (-2p_{10} - p_{11} + p_{12})x^2, \\ &\quad -2p_{10}x + p_{12}x^2) \\ &= p_{10}(1, 1 - x - 2x^2, -2x) + p_{11}(x, x - x^2, 0) + p_{12}(x^2, x^2, x^2). \end{aligned}$$

The rests of the three particular forms are equal to $4x^8 - 6x^9 + \dots$, $2x^8 + 2x^9 + \dots$ and $x^8 + x^9 + \dots$, respectively. The coefficients of x^8 and x^9 in $R\Phi(2, 2, 2)(x)$ vanish if $4p_{10} + 2p_{11} + p_{12} = 0$ and $-6p_{10} + 2p_{11} + p_{12} = 0$. Then $p_{10} = 0$ and all components of $\Phi(2, 2, 2)$ vanish for $x = 0$.

Theorem 4.5. The optimal form $\Phi_{\text{opt}}(L)$ exists and does not exist iff $\omega(L_1) < \omega(L) - 1$ and $\omega(L_1) = \omega(L) - 1$, respectively.

5. Block structure

According to Definition 1.1 a system (P_1, P_2, \dots, P_s) of polynomials not all vanishing at 0 is a Hermite–Padé form $\Phi(L)$ for every L such that

$$\partial P_i \leq l_i, \quad i = 1, 2, \dots, s, \quad \text{ord} \sum_{i=1}^s F_i P_i \geq \lambda(L), \quad (7)$$

and is not such a form for any other L . In particular, if $s = 2$ and $(F_1, F_2) = (-1, F)$ (Padé approximation) the conditions (7) determine the left upper half of a block where all the Padé approximants exist and are identical. If $s > 2$, then for an L satisfying (7) a Hermite–Padé form $\Phi(L) = (\bar{P}_1, \bar{P}_2, \dots, \bar{P}_s)$ different from $c(P_1, P_2, \dots, P_s)$ may exist. The conditions similar to (7) but concerning the \bar{P}_i 's determine as a rule another domain in the space of degrees. It has a

common part with the domain (7). Therefore, for $s > 2$ a “block” corresponding to a particular Hermite–Padé form may partly overlap another block.

Example 5.1. For the series from Example 3.2, $\Phi(0, 0, 0) = p(1, 1, 1)$ where $p \neq 0$ and $\text{ord } R\Phi(0, 0, 0) = 6$. So the conditions (7) are as follows:

$$l_j \geq 0, \quad j = 1, 2, 3, \quad l_1 + l_2 + l_3 \leq 4. \quad (8)$$

The (general) Hermite–Padé forms of some degrees obeying (8) are given below:

$$\begin{aligned} \Phi(0, 0, l), \Phi(l, 0, 0) &= p(1, 1, 1), & l &= 1, 2, 3, 4, \\ \Phi(0, l, 0) &= p(1, 1, 1) + q(0, x, 1), & l &= 1, 2, 3, 4, \\ \Phi(1, 0, l) &= p(1, 1, 1) + r(x, 0, -1 + x), & l &= 1, 2, 3, \\ \Phi(1, 1, 1), \Phi(1, 1, 2), \Phi(2, 1, 1) &= p(1, 1, 1) + q(0, x, 1) + r(x, 0, -1 + x), \\ \Phi(0, 2, 1) &= p(1, 1, 1) + (q + q'x)(0, x, 1), \\ \Phi(1, 2, 1) &= p(1, 1, 1) + (q + q'x)(0, x, 1) + r(x, 0, -1 + x). \end{aligned}$$

The parameters are such that at least one component does not vanish at 0. The degree (1, 2, 2) belongs to the right lower half of the cube defined by $0 \leq l_i \leq 4$, $i = 1, 2, 3$; cf. (8). Then we may expect that $\Phi(1, 2, 2)$ does not exist. However, it is false:

$$\Phi(1, 2, 2) = p(1, 1 - x, 0) + qx(1, 1, 1) + rx(0, x, 1), \quad p \neq 0.$$

Example 5.2. Let $s = 3$,

$$\begin{aligned} F_1(x) &= 1 - x + x^2 + x^3 + \dots, & F_2(x) &= 1 + x^3 + \dots, \\ F_3(x) &= 2 - x + x^2 + 2x^3 + \dots. \end{aligned}$$

Then every form of degree (0, 0, 0) and (1, 1, 0) is equal to $p(1, 1, -1)$, $p \neq 0$ and its rest has order ≥ 4 . On the other hand, every form of an intermediate degree (1, 0, 0) is equal to $p(1, 1, 1) + q(x, -2, 1)$, $p \neq 0$ or $q \neq 0$, and for $q \neq 0$, its rest has order 3.

It is not surprising now that a block structure of a table of all Hermite–Padé forms is generally very complex. It seems that only one theorem may be useful here. It explains when among the forms $\Phi(L)$ there exists a form of a smaller degree L^* and, in particular, when the sets of all forms of these two degrees are identical. The following ordering of degrees is used here:

$$L^* \leq L \quad \text{if } L = (l_1, l_2, \dots, l_s), \quad L^* = (l_1^*, l_2^*, \dots, l_s^*), \quad l_i^* \leq l_i, \quad i = 1, 2, \dots, s,$$

and $L^* < L$ if $L^* \leq L$, $L^* \neq L$.

Theorem 5.3. If $L^* \leq L$, then the set of all the forms $\Phi(L)$ contains a form of degree L^* iff

$$\omega(L^*) \geq \lambda(L), \quad \rho(L^*, \lambda(L)) - \rho(L_1^*, \lambda(L) - 1) < \lambda(L^*) - \lambda(L_1^*).$$

If $L^* \leq \hat{L} \leq L$, then every system (P_1, P_2, \dots, P_s) , being simultaneously a form of degree L^* and of degree L , is a form of degree \hat{L} .

The set of all the forms $\Phi(L)$ is identical with the set of all the forms $\Phi(L^*)$ iff

$$\rho(L, \lambda(L)) - \rho(L^*, \lambda(L)) = \lambda(L) - \lambda(L^*), \quad \rho(L^*, \lambda(L)) = \rho(L^*, \lambda(L^*)).$$

Let us consider now optimal Hermite–Padé forms in the sense of Definition 4.2. Such a form of a fixed degree either exists and is unique (up to a constant factor) or does not exist. Thus the table of optimal forms is a priori simpler than the table of sets of all the forms. However, it is easy to see that the strange phenomena shown above can appear also in the “optimal” table. Nevertheless, one can prove some positive facts on it.

Theorem 5.4. *If $\Phi_{\text{opt}}(L)$ exists and is equal to (P_1, P_2, \dots, P_s) , then it is also optimal for every degree L^* such that $(\partial P_1, \partial P_2, \dots, \partial P_s) \leq L^* \leq L$.*

The structure of a table of optimal forms depends on the numbers $\omega(L)$ from Lemma 1.6.

Theorem 5.5. *If $l_i \leq 0$, $i = 1, 2, \dots, s$, then $\Phi_{\text{opt}}(L)$ exists. If there exists such a j that $l_j > 0$ and if $\omega(L_1) < \omega(L) - 1$, then $\Phi_{\text{opt}}(L) = \Phi_{\text{opt}}(L^*)$, provided that $L^* < L$, $\omega(L^*) = \omega(L)$.*

If, in addition, $\omega(L^) < \omega(L)$ for every $L^* < L$, then $\Phi_{\text{opt}}(L) = (P_1, P_2, \dots, P_s)$, where $\partial P_i = l_i$, $i = 1, 2, \dots, s$.*

Let us apply Theorem 5.5 only in order to verify whether $\Phi_{\text{opt}}(L)$ for a fixed L is equal or not to the optimal form of a smaller degree. Then it suffices to find $\omega(L)$ and the numbers

$$\omega(l_1, \dots, l_{j-1}, l_j - 1, l_{j+1}, \dots, l_s) \quad \text{for every } j \text{ such that } l_j \geq 0. \quad (9)$$

In fact, if $\Phi_{\text{opt}}(L) = \Phi_{\text{opt}}(\tilde{L})$ for some $\tilde{L} < L$, then in virtue of Theorem 5.4 this is also the optimal form of degree $(l_1, \dots, l_{j-1}, l_j - 1, l_{j+1}, \dots, l_s)$ for every j such that $\tilde{l}_j < l_j$. Thus the form $\Phi_{\text{opt}}(L)$ is different from every $\Phi_{\text{opt}}(L^*)$ for $L^* < L$ iff all the numbers (9) are less than $\omega(L)$.

References

- [1] G.A. Baker Jr. and P. Graves-Morris, *Padé Approximants. Part II: Extensions and Applications*, Encyclopedia Math. Appl. **14** (Addison-Wesley, Reading, MA, 1981).
- [2] S. Paszkowski, Recurrence relations in Padé–Hermite approximation, *J. Comput. Appl. Math.* **19** (1) (1987) 99–107.